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PROOF OF A CONJECTURE OF ERDŐS ON TRIANGLES IN SET-SYSTEMS

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A triangle is a family of three sets A,B,C such that $A\cap B$, $B\cap C$, $C\cap A$ are each nonempty, and $A\cap B\cap C=\emptyset$. Let $\mathcal{A}$ be a family of r-element subsets of an n-element set, containing no triangle. Our main result implies that for $r\ge 3$ and $n\ge 3r/2$, we have $|\mathcal{A}|\le\binom{n-1}{r-1}$. This settles a longstanding conjecture of Erdős [7], by improving on earlier results of Bermond, Chvátal, Frankl, and Füredi. We also show that equality holds if and only if $\mathcal{A}$ consists of all r-element subsets containing a fixed element.

Analogous results are obtained for nonuniform families.

1. Introduction

Throughout this paper, X is an n-element set. For any nonnegative integer r, we write $X^{(r)}$ for the family of all r-element subsets of X. Define $X^{(\le r)}=\cup_{0\le i\le r}X^{(i)}$ and $X^{(\ge r)}=\cup_{r\le i\le n}X^{(i)}$. For $\mathcal{A}\subset X^{(\le n)}$ and $x\in X$, we let $\mathcal{A}_x=\{A\in\mathcal{A}: x\in A\}$.

A *triangle* is a family of three sets A,B,C such that $A\cap B$, $B\cap C$, $C\cap A$ are each nonempty, and $A\cap B\cap C=\emptyset$. Let $f(r,n)$ denote the maximum size of a family $\mathcal{A}\subset X^{(r)}$ containing no triangle. A special case of Turán's theorem (proved by Mantel) implies that $f(2,n)=\lfloor n^2/4\rfloor$. Motivated by this result, Erdős [7] asked for the determination of $f(r,n)$ for $r>2$, and conjectured that

$$f(r,n)=\binom{n-1}{r-1}\qquad\text{for}\qquad n\ge 3r/2. \tag{1}$$

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(Actually, in [7] it is stated more as a question, and $n \geq 3r/2$ is not explicitly mentioned, but later, e.g., in [3,10], (1) is referred to as a conjecture of Erdős'.)

This conjecture attracted quite a few researchers. It was proved by Chvátal [3] for $r = 3$. In fact, he proved the more general statement that if $n \geq r+2 \geq 5$, $\mathcal{A} \subset X^{(r)}$, and $|\mathcal{A}| > \binom{n-1}{r-1}$, then $\mathcal{A}$ contains r sets $A_1, \ldots, A_r$ such that every $r-1$ of them have nonempty intersection, but $\cap_i A_i = \emptyset$. This configuration is also called an $(r-1)$-dimensional simplex. Chvátal generalized (1) as follows.

Conjecture 1 (Chvátal). Let $r \geq d+1 \geq 3$, $n \geq r(d+1)/d$ and $\mathcal{A} \subset X^{(r)}$. If $\mathcal{A}$ contains no d-dimensional simplex, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. Equality holds only when $\mathcal{A} = X_x^{(r)}$, for some $x \in X$.

Recently Csákány and Kahn [6] gave a different proof of the $r = 3$ case of (1) using Homology theory. Frankl [9] settled (1) for $3r/2 \leq n \leq 2r$, and then Bermond and Frankl [2] proved (1) for infinitely many n, r, where $n < r^2$. About five years later, Frankl [10] settled (1) for $n > n_0(r)$, where $n_0(r)$ is an unspecified but exponentially growing function of r. In 1987, Frankl and Füredi [11] proved Conjecture 1 for $n > n_0(r)$. Frankl [10] had earlier verified Conjecture 1 for $(d+1)r/d \leq n < 2r$, using Katona's permutation method. Thus both (1) and Conjecture 1 remained open in the range $2r \leq n < n_0(r)$, where $n_0(r)$ is exponential in r. Also, the uniqueness of the extremal configuration remained open for $3r/2 \leq n < n_0(r)$ in both (1) and Conjecture 1.

Our main result settles (1) for all $n \geq 3r/2$ while also characterizing the extremal examples. A *non-trivial intersecting family* of size $d+1$ is a family of $d+1$ distinct sets $A_1, \ldots, A_{d+1}$ that have pairwise nonempty intersection, but $\cap_i A_i = \emptyset$.

Theorem 2. *Let $r \geq d+1 \geq 3$ and $n \geq (d+1)r/d$. Suppose that $\mathcal{A} \subset X^{(r)}$ contains no non-trivial intersecting family of size $d+1$. Then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. Equality holds if and only if $\mathcal{A} = X_x^{(r)}$ for some $x \in X$.*

Note that the special case $d = 2$ above implies (1). Every d-dimensional simplex is a non-trivial intersecting family of size $d+1$, and in this sense Theorem 2 can be thought of as a solution to a weakening of Conjecture 1.

A hypergraph $\mathcal{F}$ satisfies H_d, the Helly property of order d, if every subfamily of $\mathcal{F}$ with empty intersection contains a subcollection of at most d sets with empty intersection. A related problem is to determine the maximum size of an $\mathcal{F} \in X^{(r)}$ that satisfies H_d. Theorem 2 implies that for $d = 2$, such an $\mathcal{F}$ satisfies $|\mathcal{F}| \leq \binom{n-1}{r-1}$, however, stronger results for this problem were obtained by several authors (see Bollobás and Duchet [4,5], Tuza [15, 16], and Mulder [13]).

The proof of Theorem 2 actually works for d slightly larger than r as well. Theorem 2 is not valid when $r=3$ and $d\geq 10$ however, as the next result attests (see Section 4):

Theorem 3. *Let $\mathcal{A}\subset X^{(3)}$ contain no non-trivial intersecting family of size $d+1\geq 8$. Then*

$$|\mathcal{A}| \leq \left(\left\lfloor\frac{(d+2)}{3}\right\rfloor^{-1} + \left\lfloor\frac{(d+3)}{3}\right\rfloor^{-1} + \left\lfloor\frac{(d+4)}{3}\right\rfloor^{-1}\right)^{-1}\binom{n}{2}$$
$$\leq \frac{1}{3}\left(\left\lceil\frac{d}{3}\right\rceil + \frac{1}{d+3}\right)\binom{n}{2}.$$

Furthermore, for $d+1\geq 11$ and infinitely many n, there exists such a family $\mathcal{A}$ with $|\mathcal{A}|\geq(\frac{1}{3}\lceil\frac{d}{3}\rceil-\frac{1}{3})\binom{n}{2}$.

We conjecture that for $r\geq 4$ and n sufficiently large, the phenomenon exhibited by Theorem 3 does not arise:

Conjecture 4. Let $r\geq 4$, $d\geq 2$, and let $\mathcal{A}\subset X^{(r)}$ contain no non-trivial intersecting family of size $d+1$. Then, provided n is sufficiently large, $|\mathcal{A}|\leq\binom{n-1}{r-1}$ with equality if and only if $\mathcal{A}=X_x^{(r)}$ for some $x\in X$.

The following table summarizes the above results for $r=3$:

Size	Lower Bound	Upper Bound
$3\leq d+1\leq 7$	$\binom{n-1}{2}$	$\binom{n-1}{2}$
$d+1=8$	$\binom{n-1}{2}$	$\binom{n}{2}$
$d+1=9$	$\binom{n-1}{2}$	$\frac{12}{11}\binom{n}{2}$
$d+1=10$	$\binom{n-1}{2}$	$\frac{6}{5}\binom{n}{2}$
$d+1\geq 11$	$\frac{1}{3}(\lceil\frac{d}{3}\rceil-1)\binom{n}{2}$	$\frac{1}{3}(\lceil\frac{d}{3}\rceil+\frac{1}{d+3})\binom{n-1}{2}$

It would be interesting to determine the exact bounds for $d+1\geq 11$. In the course of the proof of the lower bound in Theorem 3, it is proved that a Steiner $(n,3,k–1)$-system, when it exists, contains no non-trivial intersecting family of size $3k+1$ whenever $k\geq 2$. We conjecture that this is the extremal family for $r=3$ and $k\geq 2$:

Conjecture 5. Let n be sufficiently large and let $k \geq 2$. Let $\mathcal{A} \subset X^{(3)}$ contain no non-trivial intersecting family of size $3k+1$, and suppose there exists a Steiner $(n,3,k-1)$-system. Then $|\mathcal{A}| \leq \frac{1}{3}(k-1)\binom{n}{2}$, with equality if and only if $\mathcal{A}$ is such a Steiner system.

Non-uniform families. It is natural to consider these extremal problems for families that are not uniform. Perhaps the most basic statement in this context is the analogue of the Erdős–Ko–Rado Theorem.

If $\mathcal{A} \subset X^{(\leq n)}$ is intersecting, then $|\mathcal{A}| \leq 2^{n-1}$.

The non-uniform analogue of Erdős' conjecture about triangles in uniform families was asked by Erdős and proved by Milner [7].

Theorem 6 (Milner). *Suppose that $\mathcal{A} \subset X^{(\leq n)}$ is triangle free. Then $|\mathcal{A}| \leq 2^{n-1}+n$.*

Since Milner's proof has not been published, we give our own short proof of this result (see also Lossers [12]). Our proof also yields that equality holds if and only if $\mathcal{A} = X_x^{(\geq 2)} \cup X^{(1)} \cup \{\emptyset\}$ for some $x \in X$; this fact seems not to have been mentioned in the previous literature. We also prove the non-uniform analogue of Theorem 2 (see Section 4).

Theorem 7. *Let $d \geq 2$ and $n > \log_2 d + \log_2 \log_2 d + 2$. Suppose that $\mathcal{A} \subset X^{(\leq n)}$ contains no non-trivial intersecting family of size $d+1$. Then $|\mathcal{A}| \leq 2^{n-1}+n$. Equality holds if and only if $\mathcal{A} = X_x^{(\geq 2)} \cup X^{(1)} \cup \{\emptyset\}$ for some $x \in X$.*

If $n \leq \lfloor \log_2 d \rfloor$, then trivially the bound $|\mathcal{A}| \leq 2^{n-1}+n$ in Theorem 7 does not hold. It can be shown that this remains true for $\lfloor \log_2 d \rfloor + 1$ and $\lfloor \log_2 d \rfloor + 2$. However, once $n > \lfloor \log_2 d \rfloor + \log_2 \log_2 d + 2$, Theorem 7 applies. It would be interesting to determine if the $\log_2 \log_2 d$ term in Theorem 7 can be replaced by an absolute constant.

2. Proof of Theorem 2

We use the notation $[n] = \{1, \ldots, n\}$ and $[a,b] = \{a, a+1, \ldots, b-1, b\}$. Let $\mathcal{A}$ be a family of r-sets with $|\mathcal{A}| \geq \binom{n-1}{r-1}$, containing no non-trivial intersecting family of size $d+1$. We prove that $\mathcal{A}$ consists of all sets containing a fixed element of X. The proof, for $n \geq (d+1)r/d$, is split into three parts;

Part I	$n < 2r$ and $n = k(n-r)+\ell$ with $k \in [2,d]$ and some $\ell \in [n-r-1]$,
Part II	$n < 2r$ and $n = k(n-r)$ with $k \in [3,d+1]$,
Part III	$n \geq 2r \geq 8$.

Note that for $(d+1)r/d \le n \le 2r-1$, there exist $k \in [2,d]$ and $\ell \in [n-r-1]$ such that $n = k(n-r)+l$ or $n=(k+1)(n-r)$. Thus Parts I and II include all these values of n.

Part I uses Katona's permutation method, Part II uses Baranyai's Theorem [1] on partitioning $X^{(r)}$ into matchings, and in Part III we proceed by induction on n. Frankl [9] established the upper bound $|\mathcal{A}| \le \binom{n-1}{r-1}$ for $(d+1)r/d \le n \le 2r-1$; however, it is substantially more difficult to establish the case of equality in Theorem 2, which we achieve in Parts I and II of our proof.

Part I: $n = k(n-r)+\ell$**.**

In this part, we consider the case $n < 2r$ and $n = k(n-r)+\ell$, for some $k \in [2,d]$ and $\ell \in [n-r-1]$. For convenience, let $X = [n]$ and fix a (cyclic) permutation π of X. Let Q_i denote the interval $\{i, i+1, \dots, i+r-1\}$ (modulo n), and let $\mathcal{A}_\pi$ denote the subfamily of $\mathcal{A}$ consisting of those sets $A \in \mathcal{A}$ such that $\pi(Q_i) = A$ for some i:

$$\mathcal{A}_\pi = \{\pi(Q_i) : \pi(Q_i) \in \mathcal{A}\}.$$

Claim 1. *Let π be any permutation. Then $|\mathcal{A}_\pi| \le r$ with equality if and only if there exists m such that*

$$\mathcal{A}_\pi = \{\pi(Q_m), \pi(Q_{m+1}), \dots, \pi(Q_{m+r-1})\}.$$

Proof. It is sufficient to prove Claim 1 for the identity permutation, since we may relabel X. Therefore $\mathcal{A}_\pi = \{Q_i : Q_i \in \mathcal{A}\}$. Without loss of generality, $Q_n \in \mathcal{A}_\pi$. For $j \in [n-r]$, let $P_j = \{i : i \equiv j \pmod{n-r}\} \cap [n]$, together with $\{n\}$ if $j \in [\ell+1, n-r]$. Thus $|P_j| \le k+1 \le d+1$. For each $j \in [n-r]$, there is an $i \in P_j$ such that $Q_i \notin \mathcal{A}_\pi$, otherwise $\bigcap_{i \in P_j} Q_i = \emptyset$. Thus $Q_i \notin \mathcal{A}_\pi$ for at least $n-r$ values of i, so $|\mathcal{A}_\pi| \le r$.

Equality holds only if there is a unique x_j such that $Q_{x_j(n-r)+j} \notin \mathcal{A}_\pi$ for all $j \in [n-r]$. We now show $x_1 \ge x_2 \ge \ldots \ge x_{n-r} \ge x_1 - 1$. Let us illustrate the proof of this fact using Figures 1 and 2 below, where $y_j = x_j(n-r)+j$, and the box (i,j) represents the integer $(i-1)(n-r)+j$:

If $x_j < x_{j+1}$ for some $j \in [\ell]$, then, since $\ell \le n-r-1$, the intersection of the $k+1$ intervals $Q_{(i-1)(n-r)+j}$, where (i,j) is a shaded box in Figure 1, is empty (this is the only place in Part I where we use $\ell \le n-r-1$; the case $\ell = n-r-1$ is the content of Part II). This contradiction implies that $x_j \ge x_{j+1}$. In a similar way, $x_j \ge x_{j+1}$ for $j \in [\ell+1, n-r]$, using $Q_n \in \mathcal{A}_\pi$. Finally, if $x_{n-r} < x_1 - 1$, then the intersection of the intervals $Q_{(i-1)(n-r)+j}$, with (i,j) a shaded box in Figure 2, is empty, a contradiction. This proves that $\mathcal{A}_\pi$ has the required form. ∎

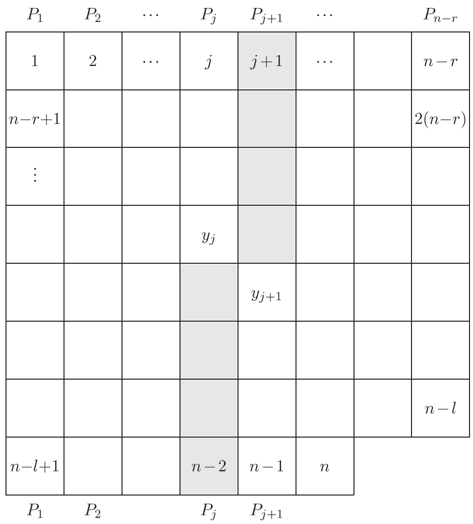

Fig. 1.

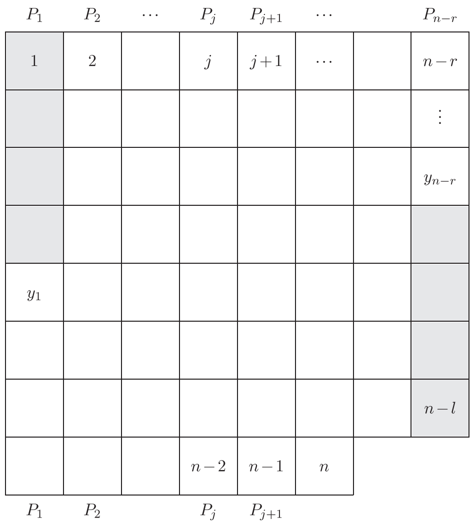

Fig. 2.

Without loss of generality, we assume that for the identity permutation ι, $\mathcal{A}_\iota = \{Q_1, Q_2, \ldots, Q_r\}$.

Claim 2. *For each permutation* π, $\mathcal{A}_\pi = \{\pi(Q_1), \pi(Q_2), \ldots, \pi(Q_r)\}$.

Proof. Each permutation π of $X \backslash \{r\}$ is a product of transpositions. Therefore it suffices to show that if τ is a transposition in which r is a fixed point, then

$$\mathcal{A}_\tau = \{\tau(Q_1), \tau(Q_2), \ldots, \tau(Q_r)\}.$$

Suppose that τ transposes t and $t+1$, with $r \notin \{t, t+1\}$. Then Claim 1 implies that $\mathcal{A}_\tau = \{\tau(Q_m), \tau(Q_{m+1}), \ldots, \tau(Q_{m+r-1})\}$ for some $m \in [n]$. We show below that $m = 1$.

Case 1. $n \notin \{t, t+1\}$: Here $\tau(Q_1) = [r] = Q_1 \in \mathcal{A}$, and $\tau(Q_n) = \{1, \ldots, r-1, n\} = Q_n \notin \mathcal{A}$. Therefore $m = 1$.

Case 2. $t+1 = n$: In this case $\tau(Q_i) = Q_i \in \mathcal{A}$ for each $i \in [r] \setminus \{n-r\}$. Consequently $\tau(Q_{n-r}) \in \mathcal{A}$ as well, and therefore $m = 1$.

Case 3. $t = n$: If $n < 2r-1$, then $\tau(Q_r) = Q_r \in \mathcal{A}$ and $\tau(Q_{r+1}) = Q_{r+1} \notin \mathcal{A}$. Therefore $m = 1$. If $n = 2r-1$, then $\tau(Q_i) = Q_i \in \mathcal{A}$ for $i = 2, \ldots, r-1$. This leaves the posibilities $m = 1, 2, n$. However, $\tau(Q_{r+1}) = Q_{r+1} \notin \mathcal{A}$, and $\tau(Q_n) = Q_n \notin \mathcal{A}$. Consequently, $\{\tau(Q_1), \tau(Q_r)\} \subset \mathcal{A}$ and $m = 1$ again. ∎

We now complete Part I. For each $A \in \mathcal{A}$, there are $\frac{1}{2} r!(n-r)!$ families $\mathcal{A}_\pi$ containing A. The total number of cyclic permutations of X is $(n-1)!/2$.

By Claim 1, $|\mathcal{A}_\pi| \le r$ and therefore

$$\tfrac{1}{2}r!(n-r)!|\mathcal{A}| \le \tfrac{1}{2}r(n-1)!.$$

This establishes the upper bound $|\mathcal{A}| \le \binom{n-1}{r-1}$. By Claim 1, equality holds if and only if for every cyclic permutation π of X, we have $\mathcal{A}_\pi = \{\pi(Q_1), \pi(Q_2), \dots, \pi(Q_r\}$. Set $x = r$. For any $A \subset (X\backslash\{x\})^{(r-1)}$, we may thus choose such a cyclic permutation π so that $\pi(Q_1) = A \cup \{x\}$. Therefore $A \cup \{x\} \in \mathcal{A}$, and $\mathcal{A} = X_x^{(r)}$ is the required family.

Part II: $n = k(n-r)$**.**

The argument here is different to that of Part I; we use a result of Baranyai [1], stating that the family $X^{(r)}$ may be partitioned into perfect matchings when r divides n. This result is only needed for the characterization of the extremal family $\mathcal{A}$. Recall that $\overline{\mathcal{A}} = \{X\backslash A : A \in \mathcal{A}\}$.

Claim 3. *If* $A \in X^{(n-r)} \backslash \overline{\mathcal{A}}$*, then* $(X\backslash A)^{(n-r)} \subset \overline{\mathcal{A}}$*.*

Proof. Pick $A' \in (X\backslash A)^{(n-r)}$. We will show that $A' \in \overline{\mathcal{A}}$. By Baranyai's Theorem, there is a partition of $X^{(n-r)}$ into perfect matchings $\mathcal{M}_1, \dots, \mathcal{M}_t$ of size k, where $t = \frac{1}{k}\binom{n}{n-r}$. By relabelling X if necessary, we may assume that $\mathcal{M}_1 \supset \{A, A'\}$. Since $\overline{\mathcal{A}}$ has no perfect matching, and $n = kr/(k-1)$,

$$|\overline{\mathcal{A}}| \le (k-1)t = \frac{k-1}{k}\binom{n}{n-r} = \frac{k-1}{k}\binom{n}{r} = \frac{k-1}{k}\frac{n}{r}\binom{n-1}{r-1} = \binom{n-1}{r-1}.$$

Therefore $|\mathcal{A}| = |\overline{\mathcal{A}}| = \binom{n-1}{r-1}$, and $|\overline{\mathcal{A}} \cap \mathcal{M}_i| = k-1$ for all i. Since $\mathcal{M}_1 \supset \{A, A'\}$ and $A \notin \overline{\mathcal{A}}$, we must have $A' \in \overline{\mathcal{A}}$. Therefore Claim 3 is verified. ∎

We now complete the proof of Theorem 2 for $n = k(n-r)$. Let $\mathcal{B} = X^{(n-r)} \backslash \overline{\mathcal{A}}$. Then $n(\mathcal{B}) = \frac{k}{k-1}r \ge 2(n-r)$ as $k \ge 2$ and $n < 2r$. Furthermore, $\mathcal{B}$ is an intersecting family, by Claim 3, and $|\mathcal{B}| = \binom{n}{n-r} - |\overline{\mathcal{A}}| = \binom{n-1}{n-r-1}$. By Theorem 2, $\mathcal{B} = X_x^{(n-r)}$ for some $x \in X$. This shows that $\mathcal{A} = X_x^{(r)}$, and Part II is complete.

Part III: $n \ge 2r$**.**

Throughout Part III, we assume $r \ge 4$. Addition of technical details in Claim 3 in the proof below accommodates the case $r = 3$. However, a short proof in this case was presented by weight counting techniques in Frankl and Füredi [11], which we revisit in Section 5.

We need the following notations.

For $\mathcal{A} \subset X^{(r)}$, let $V(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$ and $n(\mathcal{A}) = |V(\mathcal{A})|$. For $Y \subset X$, we define $\mathcal{A} - Y = \{A \in \mathcal{A} : A \cap Y = \emptyset\}$. We also write $\overline{\mathcal{A}} = \{X \backslash A : A \in \mathcal{A}\}$. The following five definitions and the associated notations will be used repeatedly throughout the paper:

Sum of Families. The *sum of families* $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t$, denoted $\sum_i \mathcal{A}_i$, is the family of all sets in each $\mathcal{A}_i$. Note that $\sum \mathcal{A}_i$ may have repeated sets, even if none of the $\mathcal{A}_i$ have repeated sets.

Trace of a Set. The *trace of a set* Y in $\mathcal{A}$ is defined by $\mathrm{tr}(Y) = \mathrm{tr}_{\mathcal{A}}(Y) = \{A \subset X : A \cup Y \in \mathcal{A}\}$. We define $\mathrm{tr}(\mathcal{A}) = \sum_{x \in X} \mathrm{tr}(x)$.

Degree of a Set. The *edge neighborhood of a set* Y is $\Gamma(Y) = \Gamma_{\mathcal{A}}(Y) = \{A \in \mathcal{A} : A \cap Y \neq \emptyset$ and $A \neq Y\}$, and the *degree of* Y is $\deg_{\mathcal{A}}(Y) = |\Gamma_{\mathcal{A}}(Y)|$. If $Y = \{y\}$, then we write y instead of $\{y\}$, and $\deg_{\mathcal{A}}(y) = |\Gamma_{\mathcal{A}}(y)| = |\mathcal{A}_y|$.

The families $\mathcal{S}_x$ and $\mathcal{L}_x$. Let $\mathcal{A}$ be an r-uniform family of sets in X and $x \in X$. Then we define

$$\mathcal{S}_x = \{Y \in \mathrm{tr}(x) : |\mathrm{tr}(Y)| = 1\} \qquad \text{and} \qquad \mathcal{L}_x = \mathrm{tr}(x) \backslash \mathcal{S}_x.$$

We write $\mathcal{S} = \sum_{x \in X} \mathcal{S}_x$ and $\mathcal{L} = \sum_{x \in X} \mathcal{L}_x = \mathrm{tr}(\mathcal{A}) \backslash \mathcal{S}$. Note that if $A \in \mathcal{L}_x$, then there exists $y \neq x$ such that $A \in \mathcal{L}_y$.

Paths and Connectivity. A *path* in $\mathcal{A}$ is a family $\mathcal{P}$ of sets $A_1, A_2, \ldots$ such that $A_i \cap A_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Family $\mathcal{A}$ is *connected* if every pair of vertices in $V(\mathcal{A})$ is contained in some path in $\mathcal{A}$. A *component* of $\mathcal{A}$ is a maximal non-empty connected subfamily of $\mathcal{A}$.

We begin with the following simple lemma. Recall that $\mathcal{B} - S = \{A \in \mathcal{B} : A \cap S = \emptyset\}$.

Lemma 8. *Let $\mathcal{B}_0$ be a finite family of sets. Then there exist disjoint sets $S_0, S_1, \ldots, S_{t-1} \in V(\mathcal{B}_0)$, such that the families $\mathcal{B}_i = \mathcal{B}_0 - \cup_{j=0}^{i-1} S_j$ for $i = 1, \ldots, t$ satisfy*

(1) $S_i \in \mathcal{B}_i$ *and* $\deg_{\mathcal{B}_i}(S_i) < d-1$ *for every* $i < t$,
(2) $\deg_{\mathcal{B}_t}(S) \geq d-1$ *for every* $S \in \mathcal{B}_t$.

Proof. For $i \geq 0$, if there exists a $T \in \mathcal{B}_i$ with $\deg_{\mathcal{B}_i}(T) < d-1$, then set $S_i = T$. Form $\mathcal{B}_{i+1}$ and repeat for $i+1$. If there is no such T, then set $i = t$ and stop. ∎

For $n \geq 2r-1$, we proceed by induction on n. The base case $n = 2r-1$ has been proved in Part I. Suppose that $n \geq 2r$, $|\mathcal{A}| = \binom{n-1}{r-1}$, and $\mathcal{A}$ contains no non-trivial intersecting family of size $d+1$. We will prove that $\mathcal{A} = X_y^{(r)}$ for some $y \in X$. This implies that if $\mathcal{A}' \subset X^{(r)}$ contains no non-trivial intersecting family of size $d+1$, then $|\mathcal{A}'| \le \binom{n-1}{r-1}$, by the following argument: If $|\mathcal{A}'| > \binom{n-1}{r-1}$, then $\mathcal{A}'$ contains an r-set R in addition to $X_y^{(r)}$ by our assumption. Now consider the subfamily consisting of all r-sets of $X_y^{(r)}$ intersecting R as well as R itself. This is clearly a non-trivial intersecting family, and it has size

$$1 + \binom{n-1}{r-1} - \binom{n-1-r}{r-1} > 1 + r > d + 1.$$

Consequently, $|\mathcal{A}'| \le \binom{n-1}{r-1}$ as claimed.

Our approach is to show that there exists a vertex $x \in X$ with $\deg_{\mathcal{A}}(x) \le \binom{n-2}{r-2}$. Subsequently, the family $\mathcal{A} - \{x\}$ has size at least $\binom{n-1}{r-1} - \binom{n-2}{r-2} = \binom{n-2}{r-1}$. By induction, equality holds and $\mathcal{A} - \{x\} = X_y^{(r)}$ for some $y \in X$; it is easy to see that every set in $\mathcal{A}$ containing x also contains y and $\mathcal{A}$ is the required family. Let us show that $\deg(x) \le \binom{n-2}{r-2}$ if $|\mathcal{L}_x|$ is a maximum.

Claim 1. $|\mathcal{L}_x| > \binom{n-3}{r-2}$.

Proof. Note that $r|\mathcal{A}| = \sum_y \deg(y) = \sum_y |\mathcal{S}_y| + \sum_y |\mathcal{L}_y|$. By the choice of x, this is at most $|\mathcal{S}| + n|\mathcal{L}_x|$. Also, $\mathcal{S} \cap \mathcal{L}_x = \emptyset$, so $|\mathcal{S}| \le \binom{n}{r-1} - |\mathcal{L}_x|$. Consequently

$$(n-1)|\mathcal{L}_x| \ge r|\mathcal{A}| - \binom{n}{r-1} = r\binom{n-1}{r-1} - \binom{n}{r-1} > (n-1)\binom{n-3}{r-2},$$

where the last inequality follows from a short computation and the fact that $r \ge 4$. Dividing by $n-1$, we obtain $|\mathcal{L}_x| > \binom{n-3}{r-2}$. ∎

Applying Lemma 8 to $\mathcal{L}_x = \mathcal{B}_0$, let $(S_0, S_1, \ldots, S_{t-1})$ be the sets in $V(\mathcal{L}_x)$ satisfying (1) and (2), and let $\mathcal{B}_i$ be as in Lemma 8. Note that $\mathcal{B}_t \ne \emptyset$, since otherwise

$$(2) \qquad |\mathcal{L}_x| \le \sum_{i=0}^{t-1} (\deg_{\mathcal{B}_i}(S_i) + 1) \le t(d-1) \le \frac{n-1}{r-1}(r-2) < n-1,$$

contradicting Claim 1. Let $\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_s$ be the components of $\mathcal{B}_t$. We let $\mathcal{K}_i'$ denote the union of $\mathcal{K}_i$ and the family of all sets in $\mathcal{S}_x$ intersecting $V(\mathcal{K}_i)$.

Claim 2. *The family $\mathcal{K}_i'$ is an intersecting family.*

Proof. Suppose, for a contradiction, that $\mathcal{K}_i'$ contains disjoint sets A_0, B_0. Since $\mathcal{K}_i$ is connected, $\mathcal{K}_i \cup \{A_0, B_0\}$ is also connected. Choose a path $A_0, A_1, A_2, \ldots, B_0$ in $\mathcal{K}_i \cup \{A_0, B_0\}$ (possibly $A_2 = B_0$). Then $A_1 \in \mathcal{K}_i$. Lemma 8 part (2) implies that $\deg_{\mathcal{K}_i}(A_1) \ge d-1$, hence (if $d \ge 4$) there exist sets

$C_1, C_2, \dots, C_{d-3} \in \mathcal{K}_i \backslash \{A_0, A_2\}$ each of which intersects A_1. By definition of $\mathcal{L}_x$, there exists $y \in X \backslash \{x\}$ such that $A_1 \in \mathcal{L}_y$. Consequently,

$$\{A_0 \cup x, A_1 \cup x, A_2 \cup x, C_1 \cup x, \dots, C_{d-3} \cup x, A_1 \cup y\}$$

is a non-trivial intersecting family of size $d+1$ in $\mathcal{A}$, since $A_0 \cap A_2 = \emptyset$, a contradiction. ∎

Claim 3. *$\mathcal{L}_x = \mathcal{K}_0$ and $n(\mathcal{L}_x) \geq n-2$.*

Proof. We first show $t = s = 0$, so that $\mathcal{L}_x = \mathcal{K}_0$. For a contradiction, suppose $t > 0$ or $s > 0$. By Claim 2, $\mathcal{K}_i \subset \mathcal{K}_i'$ is an intersecting family of $(r-1)$-sets. Therefore, for $n(\mathcal{K}_i) \geq 2(r-1)$, the Erdős–Ko–Rado theorem shows $|\mathcal{K}_i| \leq \binom{n(\mathcal{K}_i)-1}{r-2} \leq \binom{n(\mathcal{K}_i)}{r-2}$. If $n(\mathcal{K}_i) \leq 2r-3$, then $|\mathcal{K}_i| \leq \binom{n(\mathcal{K}_i)}{r-1}$, and this is at most $\binom{n(\mathcal{K}_i)}{r-2}$. Since $n(\mathcal{K}_i) \geq r-1$ for $i \leq s$, convexity of binomial coefficients yields

$$\begin{aligned}\sum_{i=0}^{s} |\mathcal{K}_i| \leq \sum_{i=0}^{s} \binom{n(\mathcal{K}_i)}{r-2} &= \binom{n(\mathcal{K}_0)}{r-2} + \sum_{i=1}^{s} \binom{n(\mathcal{K}_i)}{r-2} \\ &\leq \binom{[\sum_{i=0}^{s} n(\mathcal{K}_i)] - s(r-1)}{r-2} + \sum_{i=1}^{s} \binom{r-1}{r-2}.\end{aligned}$$

Recalling that $n(\mathcal{L}_x) = \sum_{i=0}^{s} n(\mathcal{K}_i) + t(r-1)$, we obtain

$$\sum_{i=0}^{s} |\mathcal{K}_i| \leq \binom{n(\mathcal{L}_x) - s(r-1) - t(r-1)}{r-2} + s(r-1).$$

By the argument giving the first two inequalities of (2), and $d-1 \leq r-1$, we have

$$\begin{aligned}|\mathcal{L}_x| &= \sum_{i=0}^{s} |\mathcal{K}_i| + \sum_{i=0}^{t-1} (\deg_{\mathcal{B}_i}(S_i) + 1) \\ &\leq \binom{n(\mathcal{L}_x) - (s+t)(r-1)}{r-2} + s(r-1) + t(r-1).\end{aligned}$$

If $s + t \geq 1$ then, by convexity of binomial coefficients,

$$|\mathcal{L}_x| \leq \binom{n(\mathcal{L}_x) - (r-1)}{r-2} + (r-1) \leq \binom{n-r}{r-2} + (r-1).$$

As $n \geq 2r$ and $r \geq 4$, this contradicts Claim 1. Thus $s = t = 0$, and $\mathcal{L}_x$ consists of one component, $\mathcal{K}_0$.

We now show that $n(\mathcal{L}_x) \geq n-2$. By the arguments above, $|\mathcal{K}_0| \leq \binom{n(\mathcal{K}_0)}{r-2}$. Therefore, by Claim 1, $n(\mathcal{K}_0) = n(\mathcal{L}_x) \geq n-2$. This completes the proof of Claim 3. ∎

We now complete Part III and the proof of Theorem 2, by showing that $\deg(x) \leq \binom{n-2}{r-2}$. By Claim 2, $\mathcal{K}_0'$ is an intersecting family. Since $n(\mathcal{K}_0) \geq n-2 > n-r+1$, $\mathrm{tr}(x) = \mathcal{K}_0'$ so $\mathrm{tr}(x)$ is itself an intersecting family of $(r-1)$-sets. As $n-1 \geq n(\mathcal{K}_0') \geq n(\mathcal{K}_0) \geq n-2 \geq 2(r-1)$, the Erdős–Ko–Rado theorem implies that

$$\deg(x) = |\mathcal{K}_0'| = |\mathrm{tr}(x)| \leq \binom{n-2}{r-2}.$$

This completes the proof of Theorem 2. ∎

3. Proof of Theorem 3

Part III of the proof of Theorem 2 can be extended to the case $r=3$ and $2 \leq d \leq 6$ by addition of some technical details. However, Chvátal [3] and Frankl and Füredi [11] already settled the case $r=3$ and $d=2$ so we do not consider this case here. In fact, from the proof below, it follows that for $2 \leq d \leq 6$ and $n \geq 15$, a family $\mathcal{A} \subset X^{(3)}$ containing no non-trivial intersecting family of size $d+1$ has at most $\binom{n-1}{2}$ members, with the equality as in Theorem 2.

We now prove Theorem 3, employing the weight counting methods of Frankl and Füredi.

Proof of Theorem 3. Let $\mathcal{A} \subset X^{(3)}$ and suppose $\mathcal{A}$ contains no non-trivial intersecting family of size $d+1$. Following Frankl and Füredi, the *weight of a set* $A \in \mathcal{A}$ is defined by

$$\omega(A) = \sum_{\{x,y\} \subset A} \frac{1}{|\mathrm{tr}\{x,y\}|}.$$

Then

$$\begin{aligned}\sum_{A \in \mathcal{A}} \omega(A) &= \sum_{A \in \mathcal{A}} \sum_{\{x,y\} \subset A} \frac{1}{|\mathrm{tr}(x,y)|} \\ &\leq \sum_{\{x,y\} \in X} \sum_{\substack{A \in \mathcal{A} \\ \{x,y\} \in A}} \frac{1}{|\mathrm{tr}(x,y)|} \leq \sum_{\{x,y\} \in X} 1 = \binom{n}{2}.\end{aligned}$$

Equality holds if and only if every pair in X is contained in some set in $\mathcal{A}$.

As $\mathcal{A}$ contains no non-trivial intersecting family of size $d+1$, $|\text{tr}\{x,y\}|=1$ for some $\{x,y\}\in A$ or $\sum_{x,y\in A}|\text{tr}\{x,y\}|\leq d+2$. This implies that for all $A\in\mathcal{A}$,

$$\omega(A) \geq \min\Big\{1+\tfrac{2}{n-2}, \left\lfloor\tfrac{(d+2)}{3}\right\rfloor^{-1}+\left\lfloor\tfrac{(d+3)}{3}\right\rfloor^{-1}+\left\lfloor\tfrac{(d+4)}{3}\right\rfloor^{-1}\Big\}.$$

For $d\geq 7$, the second term is smaller (in fact, less than 1). Therefore

$$\sum_{A\in\mathcal{A}}\omega(A) \geq \Big(\left\lfloor\tfrac{(d+2)}{3}\right\rfloor^{-1}+\left\lfloor\tfrac{(d+3)}{3}\right\rfloor^{-1}+\left\lfloor\tfrac{(d+4)}{3}\right\rfloor^{-1}\Big)|\mathcal{A}|.$$

Together with $\sum_{A\in\mathcal{A}}\omega(A)\leq\binom{n}{2}$, this gives the upper bound on $|\mathcal{A}|$ in Theorem 3, which is for $d\geq 7$.

For the lower bound in Theorem 3, it suffices to show that every non-trivial intersecting family of size $d+1\geq 11$ contains a pair in at least $\lceil\frac{d}{3}\rceil$ of its edges. Then a Steiner $(n,3,\lceil\frac{d}{3}\rceil-1)$-system, for those n for which such a structure exists, does not contain such an intersecting family.

Lemma 9. *Let $\mathcal{F}\subset X^{(3)}$ be a non-trivial intersecting family with $|\mathcal{F}|\geq 11$. Then there exist distinct elements $x,y\in X$ such that*

$$|\text{tr}_{\mathcal{F}}\{x,y\}| \geq \frac{1}{3}(|\mathcal{F}|-1).$$

Proof. For $a,b\in X$, we let $d(a,b)=|\text{tr}_{\mathcal{F}}\{a,b\}|$. First suppose that there exist $x,y\in X$ with $d(x,y)\geq 3$. Now let $u,v,w\in\text{tr}_{\mathcal{F}}\{x,y\}$. Throughout the proof, we assume $d(x,y)\leq\lceil\frac{1}{3}(|\mathcal{F}|-1)\rceil-1$, otherwise we are done. Let $L_x=\text{tr}_{\mathcal{F}}(x)\backslash\text{tr}_{\mathcal{F}}(y)$ and let $L_y=\text{tr}_{\mathcal{F}}(y)\backslash\text{tr}_{\mathcal{F}}(x)$. Since $\mathcal{F}$ is an intersecting family,

$(*)$ $\qquad A\cap B\neq\emptyset$ for every $A\in L_x$ and $B\in L_y$.

Case 1. *L_x contains a matching of size three.*

In this case, L_x consists of three stars with distinct centers in X. By $(*)$, every pair in L_y intersects all three centers. This implies $L_y=\emptyset$. As $\mathcal{F}$ is a non-trivial intersecting family, there is a triple in $\mathcal{F}$ disjoint from x. Since $L_y=\emptyset$ and $d(x,y)\geq 3$, this triple must be $\{u,v,w\}$ and $d(x,y)=3$. Since $\mathcal{F}$ is intersecting, the centers of the three stars must also be u,v,w. Now every $F\in\mathcal{F}-\{y\}$ with $F\neq\{u,v,w\}$ contains x. Therefore, assuming $d(x,u)\geq d(x,v)\geq d(x,w)$, we find

$$d(x,u)\geq\frac{1}{3}(|\mathcal{F}|-4)+1=\frac{1}{3}(|\mathcal{F}|-1).$$

This completes the proof in Case 1.

Case 2. *L_x and L_y contain no matching of size three.*

It is not hard to see by $(*)$ that $|L_x|+|L_y| \le 2(\lceil \frac{1}{3}(|\mathcal{F}|-1)\rceil -1)+1$, with equality if and only if L_x consists of a pair of stars of size $\lceil(|\mathcal{F}|-1)/3\rceil-1$ with distinct centers a,b and L_y consists of the pair $\{a,b\}$. Then $|\mathcal{F}|-|L_x|-|L_y|-d(x,y)\ge 2$, unless $|\mathcal{F}|=3k+2$, $k\ge 3$ and L_x and L_y are as described above. By $(*)$, any triple in $\mathcal{F}-\{x,y\}$ contains u,v and w. This shows $|\mathcal{F}-\{x,y\}|=1$, and therefore $|\mathcal{F}|=3k+2$, $k\ge 3$. In this case, $\{u,v,w\}\in\mathcal{F}$ and $a,b\in\{u,v,w\}$, since $\mathcal{F}$ is intersecting. Therefore $d(a,x)\ge 4$ (and also $d(b,x)\ge 4$), completing the proof in Case 2.

If every pair $a,b\in X$ has $d(a,b)\le 2$, then the arguments in Cases 1 and 2 still apply to give a contradiction with $|\mathcal{F}|\ge 11$, since in this case $|\mathcal{F}|=8$. This completes the proof of the Lemma. ∎

4. Proof of Theorem 7

Let $\mathcal{A}$ be a family of subsets of X containing no non-trivial intersecting family of size $d+1$. We prove Theorem 7 by showing that $\mathcal{A}'=\mathcal{A}\cap X^{(\ge 2)}$ has size at most $2^{n-1}-1$, with equality if and only if $\mathcal{A}'=X_x^{(\ge 2)}$ for some $x\in X$. Theorem 7 is proved in two parts. Part I deals with the case $d=2$, by induction on $n\ge 1$. In Part II, we use Part I to prove Theorem 7 for $d\ge 3$.

Part I: $d=2$

Theorem 7 is easily verified for $n\le 3$. Now let $n\ge 4$ and $w\in X$.

Case 1. For every partition of $X\backslash\{w\}$ into two nonempty sets Y and Z, there exists a set $A\in\mathcal{A}'-\{w\}$ such that $A\cap Y\ne\emptyset$ and $A\cap Z\ne\emptyset$. Then, for each partition of $X\backslash\{w\}$ into sets Y and Z, either $Y\cup\{w\}\notin\mathcal{A}$ or $Z\cup\{w\}\notin\mathcal{A}'$ – otherwise $\mathcal{A}'$ contains a triangle. Therefore $\deg_{\mathcal{A}'}(w)\le 2^{n-2}$. By induction, $\mathcal{A}''=\mathcal{A}'-\{w\}$ has size at most $2^{n-2}-1$, with equality if and only if $\mathcal{A}''=(X\backslash\{w\})_x^{(\ge 2)}$ for some $x\in X-\{w\}$. Thus

$$|\mathcal{A}'| = \deg_{\mathcal{A}'}(w) + |\mathcal{A}''| \le 2^{n-2} + (2^{n-2}-1) = 2^{n-1}-1.$$

Now suppose that equality holds above. We will show that every set in $\mathcal{A}'$ containing w also contains x. Suppose on the contrary that $w\in S\in\mathcal{A}'$ and $x\notin S$. Among all such S, choose the one of minimum size, call it S_0. Let T be another set containing w. By the choice of S_0, either $T\supset S_0$, or there exist $t\in T-S$, and $s\in S-T$ (possibly $t=x$). In the latter case, $\{x,s,t\},S,T$ form a triangle (replace $\{x,s,t\}$ by $\{s,t\}$ if $t=x$). We may therefore assume that every set in $\mathcal{A}'$ containing w also contains S_0. Hence

$2^{n-2} = \deg_{\mathcal{A}'}(w) \le 2^{n-|S_0|-1}$ from which we conclude that $S_0 = \{s\}$, and $E \cup \{w\} \in \mathcal{A}'$ for every $E \subset X \backslash \{w, s\}$. Since $|X| \ge 4$, there exist distinct a, b for which $\{w, s, a\}$ and $\{w, s, b\}$ lie in $\mathcal{A}'$. Together with $\{x, a, b\}$ (or just $\{a, b\}$ if $a = x$ or $b = x$) this once again forms a triangle.

Case 2. There exists a partition of $X \backslash \{w\}$ into two nonempty sets Y and Z such that no member of $\mathcal{A}'$ in $X \backslash \{w\}$ contains an element of both Y and Z. By induction, at most $2^{|Y|} - 1$ elements of $\mathcal{A}'$ are contained in $Y \cup \{w\}$, and similarly for Z. The number of sets which contain an element of Y and an element of Z is, by the choice of Y and Z, at most $2^{n-1} - 1 - (2^{|Y|} - 1) - (2^{|Z|} - 1)$. Therefore

$$|\mathcal{A}'| \le (2^{|Y|} - 1) + (2^{|Z|} - 1) + (2^{n-1} - 1 - (2^{|Y|} - 1) - (2^{|Z|} - 1)) = 2^{n-1} - 1.$$

If equality holds, then by induction there exist $y \in Y \cup \{w\}, z \in Z \cup \{w\}$ with $(Y \cup \{w\})_y^{(\ge 2)} \cup (Z \cup \{w\})_z^{(\ge 2)} \subset \mathcal{A}'$. Since $|X| \ge 4$, we may assume by symmetry that $|Y| \ge 2$. We next show that $y = w$. Observe that for every set $S \subset X \backslash \{w\}$ containing an element of both Y and Z, we have $S \cup \{w\} \in \mathcal{A}'$. If $y \ne w$, then $\{y, a\} \in \mathcal{A}'$ for some $a \in Y \backslash \{y\}$. The set $\{y, a\}$ together with $\{y, w\}$ and $\{w, a, b\}$ for some $b \in Z$ forms a triangle. Consequently $y = w$, and $z = w$ as well unless $Z = \{z\}$. But in this case $(Z \cup \{w\})_z^{(\ge 2)} = (Z \cup \{w\})_w^{(\ge 2)}$, therefore $\mathcal{A}' = X_w^{(\ge 2)}$.

Part II: $d \ge 3$

Define a function f on the positive integers by $f(1) = f(2) = f(3) = 1$, and for $n \ge 4$,

$$(*) \qquad f(n) = \max\{0, f(n-3) + d - 2^{n-4}\}.$$

It is easy to see that if $n \ge 4$, and $f(n) \ge 0$, then

$$f(n) = 1 + \left(\left\lceil \frac{n}{3} \right\rceil - 1 \right) d - \sum_{i=0}^{\lfloor (n-4)/3 \rfloor} 2^{n-4-3i}.$$

Set $n_d = \log_2 d + \log_2 \log_2 d + 2$. An easy calculation now shows that $f(n) = 0$ whenever $n \ge n_d$ and $f(n) > f(n-3) + d - 2^{n-4}$ when $n > n_d$.

In this part of the proof, we proceed by induction on $n \ge 1$, with the following hypothesis: Let $\mathcal{A}' \subset X^{(\ge 2)}$ contain no non-trivial intersecting family of size $d+1$. Then $|\mathcal{A}'| \le 2^{n-1} - 1 + f(n)$.

For $n \le 3$, the result is true as

$$|\mathcal{A}'| \le |X^{(\ge 2)}| = 2^n - n - 1 \le 2^{n-1} - 1 + f(n).$$

Now suppose that $n \geq 4$. By Part I, we may assume $\mathcal{A}'$ contains a triangle $\mathcal{F} = \{F_1, F_2, F_3\}$, otherwise the proof is complete.

Let x, y, z be elements in $F_1 \cap F_2$, $F_2 \cap F_3$ and $F_3 \cap F_1$ respectively. Then at most d sets in $\mathcal{A}'$ intersect $\{x, y, z\}$ in at least two points, otherwise $\mathcal{F}$ together with another $d-2$ of these sets forms a non-trivial intersecting family of size $d+1$. The total number of sets in $\mathcal{A}'$ intersecting $\{x, y, z\}$ is therefore at most $3 \cdot 2^{n-3} + d$. Let $\mathcal{A}'' = \mathcal{A}' - \{x, y, z\}$. Then

$$|\mathcal{A}'| \leq |\mathcal{A}''| + 3 \cdot 2^{n-3} + d.$$

As $\mathcal{A}''$ contains no non-trivial family of size $d+1$, the induction hypothesis shows $|\mathcal{A}''| \leq 2^{n-4} - 1 + f(n-3)$. This gives

$$\begin{aligned} |\mathcal{A}'| &\leq 2^{n-4} - 1 + f(n-3) + 3 \cdot 2^{n-3} + d \\ &= 2^{n-1} - 1 + f(n) - (f(n) - f(n-3) - d + 2^{n-4}) \\ &\leq 2^{n-1} - 1 + f(n), \qquad (**) \end{aligned}$$

where the last inequality follows from $(*)$. By the choice of n_d, we know that $f(n) = 0$ for $n \geq n_d$, so $|\mathcal{A}'| \leq 2^{n-1} - 1$ for $n \geq n_d$, completing the proof of the upper bound in Theorem 7.

Now suppose that $|\mathcal{A}'| = 2^{n-1} - 1$ and $n > n_d$. Then the inequality $(**)$ is strict. This gives the contradiction $|\mathcal{A}'| < 2^{n-1} - 1$. Consequently $\mathcal{A}'$ contains no triangle and Part I of the proof applies to give the case of equality. ∎

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